

On Wagoner complexes

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Wagoner complexes are simplicial complexes associated to groups of Kac-Moody type. They admit interesting homotopy groups which are related to integral group homology if the root datum is of 2-spherical type. We give a general definition of Wagoner complexes, exhibit some simple properties and calculate low dimensional homotopy groups. In addition, we give a definition of affine Wagoner complexes related to groups admitting a root datum with valuation.

0 Introduction

Wagoner complexes are certain simplicial complexes first constructed by Wagoner in [Wag73]. In his original construction, they were associated to the group $GL_n(R)$ over an arbitrary ring R . Wagoner used the homotopy groups of these complexes to give a tentative definition of higher algebraic K -theory. Later Anderson, Karoubi and Wagoner proved that this definition is equivalent to Quillen's definition of K -theory.

The complexes Wagoner defined are closely related to the building associated to the general linear group. Unlike the building, the Wagoner complex is not homotopy equivalent to a bouquet of spheres, instead it has interesting homotopy and homology groups. Wagoner complexes have already been generalised to reductive F -groups by Petzold in [Pet04]. There, rational homology groups of Wagoner complexes are used to construct rational representations of reductive groups.

In this paper, we define Wagoner complexes in the context of groups of Kac-Moody type (or groups with root data), which we believe to be the appropriate level of generality. Important examples of such groups are semi-simple Lie groups and algebraic groups as well as Kac-Moody groups. All definitions are very natural and the connection to the theory of buildings is clearly visible.

We exhibit interesting substructures in Wagoner complexes analogous to apartments in buildings and present some general properties of Wagoner complexes. The main result is the following characterisation of low dimensional homotopy groups of Wagoner complexes in the case where the associated root datum is of 2-spherical type.

Theorem 0 *Let G be a group of Kac-Moody type and assume that its root datum is of 2-spherical type. Let G^\dagger be the little projective group and let \hat{G} be the associated Steinberg group in the sense of Tits. Let $\mathcal{W}(G)$ be the associated Wagoner complex.*

Then, in the generic case, we have

$$\pi_0(|\mathcal{W}(G)|) \cong G/G^\dagger.$$

If G is of non-spherical type or $\text{rank}(G) \geq 3$, we have

$$\pi_1(|\mathcal{W}(G)|) \cong \ker(\hat{G} \twoheadrightarrow G^\dagger).$$

By a recent result by Caprace, this means $\pi_1(|\mathcal{W}(G)|) \cong H_2(G^\dagger; \mathbb{Z})$ except over very small fields.

For the precise statement, see Theorem 14 and Corollary 15.

In addition to this result, we also make the connection to algebraic K -theory explicit. Finally, we give a definition of affine Wagoner complexes associated to groups admitting root data with valuations.

The structure of this text is as follows. In section 1, we define Wagoner complexes. We construct apartments and a projection onto the building for spherical Wagoner complexes in section 2. In section 3, we discuss simple properties of pairs of roots. The definition of the Steinberg group in the sense of Tits is given in section 4. We prove the main result in section 5 and show its relation to algebraic K -theory in section 6. Finally, we give the construction of affine Wagoner complexes in section 7.

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1 The definition of Wagoner complexes

Throughout this text, let (W, S) be a Coxeter system of rank at least 2 whose Coxeter diagram has no isolated nodes. Let Σ be the associated Coxeter complex. Denote by Φ the set of roots of Σ and let G be a group with a twin root datum $(G, (U_\alpha)_{\alpha \in \Phi}, H)$ of type Φ . For the precise definition of groups with root data, also known as groups of Kac-Moody type, see the book [Rém02] by Rémy. Denote the subgroup of G generated by all root groups, the *little projective group*, by G^\dagger .

We write $R_s = \{t \in \Sigma : s \subseteq t\}$ for the *residue* of s in Σ . We will say that a simplex s is *co-spherical* if the corresponding residue R_s is of spherical type. The basic ‘building blocks’ for Wagoner complexes will be the following groups.

Definition 1 For a co-spherical simplex $s \in \Sigma$ we define

$$U_s := \langle U_\alpha : R_s \subseteq \alpha \in \Phi \rangle.$$

It is well known that there is a twin building of type Φ associated to the group G , see [R  m02, 2.6.4]. By the same reference, the positive half of the twin building is a thick Moufang building Δ on which G acts strongly transitively. There is a standard apartment in the building which we identify with Σ , and there is a fundamental chamber $c_0 \in \Sigma$. Using the Moufang property of the building, we obtain:

Remark We have that $s \leq s'$ if and only if $R_s \supseteq R_{s'}$ which is equivalent to $U_s \leq U_{s'}$.

For every $s \in \Sigma$, the parabolic subgroup P_s is the stabiliser subgroup of s in G . We have $U_s \leq P_s$.

Remark The building Δ is isomorphic to the set of cosets $\{gP_s : s \subseteq c_0, g \in G\}$ of standard parabolics partially ordered by reverse inclusion, see for example [AB08, 6.29]. We will henceforth identify the building with this partially ordered set.

Generalising the construction by Wagoner in [Wag73], we define:

Definition 2 The Wagoner complex $\mathcal{W}(G)$ is the flag complex over the set of cosets $\{gU_s : s \in \Sigma, s \text{ co-spherical}, g \in G\}$ partially ordered by inclusion, which we write as follows:

$$\mathcal{W}(G) := \text{Flag}(\{gU_s : s \in \Sigma, s \text{ co-spherical}, g \in G\}).$$

We call a Wagoner complex spherical if the root datum is of spherical type.

2 Simple Properties of spherical Wagoner complexes

Although we make no direct use of this, let us examine some interesting properties of spherical Wagoner complexes for further reference. Most of these results were already observed in [Pet04] or [Wag73].

We restrict ourselves to spherical Wagoner complexes in this section, because the corresponding results are more interesting. We will comment on the general situation at the end of this section. For the rest of this section, assume that Σ is a spherical Coxeter complex.

Definition 3 We define the standard apartment of $\mathcal{W}(G)$ to be

$$\mathcal{A} := \text{Flag}(\{U_s : s \in \Sigma\}).$$

Left translates of the standard apartment will be called apartments.

A spherical Wagoner complex admits a natural projection onto the barycentric subdivision of the building, which can be described as the flag complex over the building.

Projection Consider the following map of partially ordered sets

$$\begin{aligned}\tilde{p}_\Delta : \{gU_s : s \in \Sigma, g \in G\} &\rightarrow \Delta \\ gU_s &\mapsto gs.\end{aligned}$$

This map is well-defined since $U_s \leq P_s$. In addition, it is easily seen to be order-preserving. In particular, it induces a surjective simplicial map $p_\Delta : \mathcal{W}(G) \rightarrow \text{Flag}(\Delta)$ onto the barycentric subdivision of the building Δ .

It is easy to see that the projection p_Δ maps the standard apartment \mathcal{A} bijectively onto the barycentric subdivision of the standard apartment $\text{Flag}(\Sigma)$ which justifies the terminology. In addition, the map p_Δ induces a surjective map on singular homology: by the theorem of Solomon and Tits, the singular homology of the building Δ is concentrated in top-dimension and it is generated by the apartments which are all images of apartments in $\mathcal{W}(G)$.

Right N -action There is a right action of the group N on the vertices of the Wagoner complex $\{gU_s : g \in G, s \in \Sigma\}$ by right multiplication:

$$gU_s n = gnU_{n^{-1}s}.$$

Since $U_c \cap N = \{1\}$ for any chamber c , this action is free. It induces a free right N -action on $\mathcal{W}(G)$ and we have $\mathcal{A}n = n\mathcal{A}$. As already observed in [Pet04], the projection $|\mathcal{W}(G)| \rightarrow |\mathcal{W}(G)/N|$ is a fibration with discrete fibre N . The projection p_Δ factors through $\mathcal{W}(G)/N$, since

$$\tilde{p}_\Delta(gnU_{n^{-1}s}) = gn(n^{-1}s) = gs = \tilde{p}_\Delta(gU_s).$$

If we consider a complex analogous to the Wagoner complex but replacing the groups U_s by the parabolic subgroups P_s , we obtain another simplicial complex with a right N -action. The quotient complex is isomorphic to the barycentric subdivision of the building. The following diagram summarises all these constructions.

$$\begin{array}{ccc}\mathcal{W}(G) = \text{Flag}(\{gU_s : s \in \Sigma, g \in G\}) & \xrightarrow{\quad} & \text{Flag}(\{gP_s : s \in \Sigma, g \in G\}) \\ \downarrow \text{mod } N & \searrow p_\Delta & \downarrow \text{mod } N \\ \mathcal{W}(G)/N \cong \text{Flag}(\{gU_s : s \subseteq c_0, g \in G\}) & \longrightarrow & \text{Flag}(\Delta) \cong \text{Flag}(\{gP_s : s \subseteq c_0, g \in G\})\end{array}$$

The general case In the general case for non-spherical root data, the concept of apartments and the above projection still make sense. The projection is then onto the flag complex over all co-spherical simplices of Δ , which is the simplicial complex underlying the *Davis realisation* of the building Δ as in [Dav08, 18.2]. The Davis realisation is contractible, so the induced maps on homotopy and homology groups are necessarily trivial.

3 Root intervals

Remember that a pair of roots (α, β) is called *prenilpotent*, if the intersections $\alpha \cap \beta$ and $-\alpha \cap -\beta$ both contain a chamber. For any prenilpotent pair of roots (α, β) , we write

$$\begin{aligned} [\alpha, \beta] &:= \{\gamma \in \Phi : (\alpha \cap \beta) \subseteq \gamma, (-\alpha \cap -\beta) \subseteq -\gamma\}, \\]\alpha, \beta] &:= [\alpha, \beta] \setminus \{\alpha\}, \\ [\alpha, \beta[&:= [\alpha, \beta] \setminus \{\beta\}. \end{aligned}$$

These sets of roots are always finite. For any of the three sets of roots

$$\Psi \in \{[\alpha, \beta],]\alpha, \beta], [\alpha, \beta[\},$$

we write

$$U_\Psi := \langle U_\gamma : \gamma \in \Psi \rangle.$$

Remark For spherical Coxeter systems, a pair of roots (α, β) is prenilpotent if and only if $\alpha \neq -\beta$. In this case, the root interval can be expressed simply as

$$[\alpha, \beta] = \{\gamma \in \Phi : (\alpha \cap \beta) \subseteq \gamma\}.$$

For any root $\alpha \in \Phi$, we denote by s_α the reflection at its wall $\partial\alpha$. In non-spherical Coxeter complexes, there are two possibilities for prenilpotent pairs of roots.

Lemma 4 (Proposition 3.165 and Lemma 8.45 in [AB08]) *Let (α, β) be a prenilpotent pair of roots in Φ . Then there are two possibilities*

- *The product $s_\alpha s_\beta$ has infinite order. In this case, the roots α and β are nested, that is, we have $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*
- *The product $s_\alpha s_\beta$ has finite order. Then every maximal simplex s of the intersection of the walls $\partial\alpha \cap \partial\beta$ has codimension 2 and the link $\text{lk}(s)$ is spherical. We write $\bar{\alpha} = \alpha \cap \text{lk}(s)$, $\bar{\beta} = \beta \cap \text{lk}(s)$ for the induced roots in the link. The map given by*

$$\begin{aligned} [\alpha, \beta] &\rightarrow [\bar{\alpha}, \bar{\beta}] \\ \gamma &\mapsto \gamma \cap \text{lk}(s) \end{aligned}$$

is a bijection.

This can be used to prove the following characterisation.

Lemma 5 *For any prenilpotent, non-nested pair of roots (α, β) , there are either roots α' and β' such that*

$$]\alpha, \beta] = [\alpha', \beta], \quad [\alpha, \beta[= [\alpha, \beta']$$

or $]\alpha, \beta] = \{\beta\}$ and $[\alpha, \beta[= \{\alpha\}$.

Proof If the Coxeter complex Σ is of rank two and finite, it is an ordinary $2m$ -gon for some $m \geq 2$. Its roots are paths of length m . We denote the vertices of Σ by $(x_i)_{i \in \mathbb{Z}/2m}$. This leads to an enumeration of the roots $(\alpha_i)_{i \in \mathbb{Z}/2m}$ such that every root α_i contains the vertices $\{x_i, x_{i+1}, \dots, x_{i+m}\}$. Then it is easy to see that

$$[\alpha_0, \alpha_i] = \{\alpha_0, \alpha_1, \dots, \alpha_i\}$$

for $1 \leq i \leq m-1$. This implies in particular $[\alpha_0, \alpha_1] = \{\alpha_0, \alpha_1\}$ and $] \alpha_0, \alpha_i] = [\alpha_1, \alpha_i]$ and $[\alpha_0, \alpha_i[= [\alpha_0, \alpha_{i-1}]$ for $i > 1$. Since the numbering of vertices was arbitrary, the Lemma follows for rank two finite Coxeter complexes.

If the rank of the Coxeter complex Σ is greater than two, and if (α, β) is a prenilpotent, non-nested pair of roots, by Lemma 4, we can reduce to the rank 2 case as follows. Fix a maximal simplex s in the intersection of the walls $\partial\alpha \cap \partial\beta$ and write $\bar{\alpha} = \alpha \cap \text{lk}(s)$, $\bar{\beta} = \beta \cap \text{lk}(s)$ as above. Then consider the bijection

$$\begin{aligned} [\alpha, \beta] &\rightarrow [\bar{\alpha}, \bar{\beta}] \\ \gamma &\mapsto \gamma \cap \text{lk}(s) \end{aligned}$$

Now the lemma follows by applying the finite rank two case to $\text{lk}(s)$ and using the bijection to transport the result to the ambient building. \square

Using this characterisation, we prove another lemma which describes the ‘shadows’ of root group intervals in the groups U_s .

Lemma 6 *Let (α, β) be a prenilpotent, non-nested pair of roots. Let $s \in \alpha \cap \beta$ be a co-spherical simplex. Then one of the following holds.*

- *There is a prenilpotent, non-nested pair of roots (γ, δ) such that $R_s \subset \gamma \cap \delta$ and such that $U_s \cap U_{[\alpha, \beta]} \subseteq U_{[\gamma, \delta]}$.*
- *There is a root γ containing R_s such that $U_s \cap U_{[\alpha, \beta]} \subseteq U_\gamma$.*

Proof The set of chambers in R_s is finite, and we denote the number of chambers in $R_s \setminus (\alpha \cap \beta)$ by n . If $n = 0$, then (α, β) is already a pair of roots we are looking for. Otherwise, we will replace $U_{[\alpha, \beta]}$ iteratively by smaller groups as follows, always decreasing n strictly.

If $n > 0$, then $R_s \not\subset \alpha$ or $R_s \not\subset \beta$. We fix a chamber $c \in R_s \cap \alpha \cap \beta$, and a chamber $d \in -\alpha \cap -\beta$. Denote by e the projection of d onto R_s . Since the roots $-\alpha$ and $-\beta$ are convex and $d \in -\alpha \cap -\beta$, we have $e \in -\alpha \cap R_s$ or $e \in -\beta \cap R_s$. By [R  m02, 2.2.6], the minimal gallery from c to d through e crosses one of the walls $\partial\alpha$ or $\partial\beta$ first among the walls of roots in $[\alpha, \beta]$. Assume that the gallery crosses the wall $\partial\alpha$ first. Then there is a chamber $c' \in -\alpha \cap R_s$ such that $c' \in \gamma$ for all $\gamma \in]\alpha, \beta]$.

By [AB08, Proposition 8.33], we have $U_{[\alpha, \beta]} = U_\alpha U_{] \alpha, \beta]}$. Then

$$U_s \cap U_{[\alpha, \beta]} \subseteq U_s \cap U_{] \alpha, \beta]}$$

since every non-trivial element of U_α cannot fix c' , while all elements of $U_{\lceil\alpha,\beta\rceil}$ and of U_s fix c' . Now, by Lemma 5, either there is a root α' such that $U_{\lceil\alpha,\beta\rceil} = U_{\lceil\alpha',\beta\rceil}$ or $\lceil\alpha,\beta\rceil = \{\beta\}$.

In the first case, the number of chambers n' in $R_s \setminus (\alpha' \cap \beta)$ is strictly smaller than the number of chambers in $R_s \setminus (\alpha \cap \beta)$, since the first one is a subset of the second and does not contain c' . If $n' = 0$, we are finished, otherwise we repeat the procedure.

In the second case, we have $U_{\lceil\alpha,\beta\rceil} = U_\beta$. Then either $R_s \in \beta$ and we are done, or $U_{\lceil\alpha,\beta\rceil} \cap U_s \subseteq U_s \cap U_\beta = \{1\}$, in which case we can choose an arbitrary root γ containing R_s . \square

4 The Steinberg group

From now on, we will often require the following technical condition. Denote by Π the set of simple roots of the root system Φ . For roots $\alpha, \beta \in \Pi$, we let

$$X_\alpha := \langle U_\alpha \cup U_{-\alpha} \rangle, \quad X_{\alpha,\beta} := \langle X_\alpha \cup X_\beta \rangle.$$

Following Caprace in [Cap07], we consider the following condition, which excludes the possibility that $X_{\alpha,\beta}$ modulo its centre is isomorphic to some small finite groups.

$$X_{\alpha,\beta}/Z(X_{\alpha,\beta}) \not\cong B_2(2), G_2(2), G_2(3), {}^2F_4(2) \quad \text{for all pairs } \{\alpha, \beta\} \subseteq \Pi. \quad (\text{Co}^*)$$

We will require this condition for most of the results later on. Now, following Tits in [Tit87], we introduce the Steinberg group. The definition uses the concept of the direct limit of a system of groups as in [Ser80, §1].

Definition 7 *The Steinberg group \hat{G} associated to the group G is the direct limit of the system of groups U_α and $U_{\lceil\alpha,\beta\rceil}$ with their natural inclusions for all roots α and all prenilpotent pairs of roots (α, β) .*

There is an improved characterisation of the Steinberg group due to Caprace, where we can restrict ourselves to non-nested pairs of roots.

Theorem 8 (Proposition 3.9 in [Cap07]) *Assume that the group G is of 2-spherical type and satisfies condition (Co^{*}). Then the Steinberg group \hat{G} is isomorphic to the direct limit of the system of groups U_α and $U_{\lceil\alpha,\beta\rceil}$ with their natural inclusions for all roots α and all prenilpotent, non-nested pairs of roots (α, β) .*

There is a similar characterisation for the groups U_c .

Theorem 9 (Theorem 3.6 in [Cap07]) *Assume that the group G is of 2-spherical type and satisfies condition (Co^{*}). For any chamber $c \in \Sigma$, the group U_c is the direct limit of the system of groups U_α and $U_{\lceil\alpha,\beta\rceil}$ for all roots α containing c and all prenilpotent, non-nested pairs of roots (α, β) containing c , respectively.*

We need this statement also for the groups U_s .

Lemma 10 *Assume again that G is of 2-spherical type and satisfies condition (Co*). For any co-spherical simplex $s \in \Sigma$, the group U_s is the direct limit of the system of groups U_α and $U_{[\alpha, \beta]}$ for all roots α where $R_s \subset \alpha$ and all prenilpotent, non-nested pairs of roots (α, β) with $R_s \subset (\alpha \cap \beta)$, respectively.*

Proof If $s = c$ is a chamber, this is Theorem 9. Now let $s \in \Sigma$ be any co-spherical simplex and fix a chamber c that contains s . Consider the following systems of groups

$$\begin{aligned}\mathcal{U}_c &:= \{U_\alpha : c \in \alpha\} \cup \{U_{[\alpha, \beta]} : c \in \alpha \cap \beta, (\alpha, \beta) \text{ non-nested}\} \\ \mathcal{U}_s &:= \{U_\alpha : R_s \subset \alpha\} \cup \{U_{[\alpha, \beta]} : R_s \subset \alpha \cap \beta, (\alpha, \beta) \text{ non-nested}\}\end{aligned}$$

with the canonical inclusions. Since $\mathcal{U}_s \subseteq \mathcal{U}_c$, by the universal property of $\varinjlim \mathcal{U}_s$, we obtain a homomorphism

$$\varphi : \varinjlim \mathcal{U}_s \rightarrow \varinjlim \mathcal{U}_c$$

which is the identity on all groups $U \in \mathcal{U}_s$. We have the following diagram:

$$\begin{array}{ccc} \varinjlim \mathcal{U}_c & \xrightarrow{\cong} & U_c \\ \uparrow \varphi & \nearrow & \uparrow \\ \varinjlim \mathcal{U}_s & \twoheadrightarrow & U_s \end{array}$$

where the bottom row is surjective since $U_s = \langle U : U \in \mathcal{U}_s \rangle$. It remains to show that φ is injective. Then the diagonal map is an isomorphism onto the image of U_s in U_c .

It is well known that the direct limit of a system of groups can be constructed explicitly as the free product of the involved groups modulo the relations coming from the inclusions in the system of groups, see [Ser80, §1]. Assume that φ is not injective, and fix $1 \neq g \in \varinjlim \mathcal{U}_s$ with $\varphi(g) = 1$.

By the above description, the element g can be written as a word $g = g_1 \cdots g_n$ with $g_i \in U_i \in \mathcal{U}_s$ and with $n > 0$ minimal.

If $n = 1$, then $g \in U_1 \in \mathcal{U}_s \subset \mathcal{U}_c$. But then $\varphi(g) = 1$ implies $g = 1$, since φ is the identity on U_1 .

If $n > 1$, then $\varphi(g) = \varphi(g_1) \cdots \varphi(g_n) = 1$. By the description of the direct limit $\varinjlim \mathcal{U}_c$, there is $1 \leq i < n$ such that $\varphi(g_i)$ and $\varphi(g_{i+1})$ are contained in a common group $V \in \mathcal{U}_c$, since the word $\varphi(g_1) \cdots \varphi(g_n)$ can be reduced to 1.

The group V is either a root group or a root group interval. In both cases, there is a prenilpotent, non-nested pair of roots (α, β) with $c \in \alpha \cap \beta$ such that $\varphi(g_i)$ and $\varphi(g_{i+1})$ are contained in $U_{[\alpha, \beta]}$.

But observe that $\varphi(g_i)$ and $\varphi(g_{i+1})$ are contained in U_s , so by Lemma 6, there is a group $U \in \mathcal{U}_s$ such that

$$\{\varphi(g_i), \varphi(g_{i+1})\} \subset U_{[\alpha, \beta]} \cap U_s \subseteq U$$

Hence the product $g_i g_{i+1}$ can already be formed in the group $\varinjlim \mathcal{U}_s$ and

$$g_1 \cdots g_{i-1} (g_i g_{i+1}) g_{i+2} \cdots g_n$$

is a representation of g as a word of shorter length in $\varinjlim \mathcal{U}_s$, which contradicts the minimality of n . So φ is injective. \square

All of these characterisations allow us to give a different description of the Steinberg group.

Proposition 11 *Assume that G is 2-spherical and satisfies condition (Co*). The direct limit \tilde{G} of the system of groups U_s for all co-spherical simplices $s \in \Sigma$ with respect to their natural inclusions is isomorphic to the Steinberg group \hat{G} .*

Proof The proof will be divided into two steps. In the first step, we construct a map $\hat{G} \rightarrow \tilde{G}$, the inverse map is constructed in the second step.

Step 1: Of course, $U_\alpha \leq U_s$ for any co-spherical simplex s such that $R_s \subset \alpha$. So we have an inclusion $U_\alpha \hookrightarrow U_s$ for any such simplex s . Roots are connected and chambers and panels are always co-spherical. So for any two co-spherical simplices s, s' with $R_s, R_{s'} \subset \alpha$, there is a sequence of co-spherical simplices s_1, \dots, s_n such that

$$U_s \left\{ \begin{smallmatrix} \subseteq \\ \supseteq \end{smallmatrix} \right\} U_{s_1} \left\{ \begin{smallmatrix} \subseteq \\ \supseteq \end{smallmatrix} \right\} \dots \left\{ \begin{smallmatrix} \subseteq \\ \supseteq \end{smallmatrix} \right\} U_{s_n} \left\{ \begin{smallmatrix} \subseteq \\ \supseteq \end{smallmatrix} \right\} U_{s'},$$

and the group U_α is contained in all of these. Hence the inclusion $U_\alpha \hookrightarrow \tilde{G}$, induced by any inclusion $U_\alpha \hookrightarrow U_s$ for any co-spherical simplex s with $R_s \subset \alpha$, does not depend on the choice of s . A similar argument works for the inclusions $U_{[\alpha, \beta]} \hookrightarrow U_s$ for any co-spherical simplex s such that $R_s \subset (\alpha \cap \beta)$. By the universal property of the colimit, there is hence a canonical homomorphism $\hat{G} \rightarrow \tilde{G}$ induced by these inclusions.

Step 2: By Lemma 10, every group U_s can be written as a direct limit of a subsystem of the direct system for \hat{G} , hence there is a canonical homomorphism $U_s \rightarrow \hat{G}$. Since these homomorphisms are all induced by the inclusions of root groups, they are compatible with the direct system for \tilde{G} . So there is a canonical epimorphism $\tilde{G} \rightarrow \hat{G}$.

A closer inspection yields that the composition of these maps is by construction trivial on the root groups U_α , hence they invert each other. \square

The importance of the Steinberg group for this paper comes from the following theorem by Caprace, for which we require some additional notation. For every root α write $X_\alpha := \langle U_\alpha \cap U_{-\alpha} \rangle$ and $H_\alpha = N_{X_\alpha}(U_\alpha) \cap N_{X_\alpha}(U_{-\alpha})$. For every pair of roots (α, β) let $X_{\alpha, \beta} := \langle X_\alpha \cap X_\beta \rangle$ and let $\Phi_{\alpha, \beta}$ be the rank 2 root system generated by α and β . Set $H_{\alpha, \beta} = \bigcap_{\gamma \in \Phi_{\alpha, \beta}} N_{X_{\alpha, \beta}}(U_\gamma)$. Then $(X_{\alpha, \beta}, (U_\gamma)_{\gamma \in \Phi_{\alpha, \beta}}, H_{\alpha, \beta})$ is a root datum of rank 2.

Theorem 12 (Caprace, Theorem 3.11 in [Cap07]) *Assume that the group G is of 2-spherical type and that its diagram has no isolated nodes. Suppose also that*

1. *For each $\alpha \in \Pi$, we have $[H_\alpha, U_\alpha] = U_\alpha$.*

2. For each pair of roots (α, β) in Π such that the corresponding Coxeter group elements s_α and s_β do not commute, the Steinberg group $\hat{X}_{\alpha, \beta}$ is the universal central extension of $X_{\alpha, \beta}$.

Then \hat{G} is the universal central extension of the little projective group G^\dagger .

Condition (Co*) is implied by the first condition, see the proof of Theorem 3.11 in [Cap07]. Furthermore, note that the second condition is essentially a condition on Moufang polygons. There is a complete classification of Moufang polygons satisfying this condition, see [dMT09], stating that it is almost always valid except in the case of small fields. Using [TW02, 33.10-33.17], one can calculate that the first condition is also always fulfilled except for small fields.

5 Homotopy groups of Wagoner complexes

There are various results to calculate the fundamental groups of complexes formed out of group cosets, for instance by Soulé, Brown and Abels-Holz. An equivalent point of view is to consider a group acting on a simplicial complex and to construct a presentation using stabilisers and the fundamental group of the quotient complex.

We require a very simple form of a result of this type. Unfortunately, we could not find such a simple formulation in the literature, so we include a precise statement of this result here. We then use the modern approach of complexes of groups as in [BH99] to give a simple proof.

Theorem 13 (II.12.20 in [BH99]) *Let X be a finite-dimensional simplicial complex and let a group G act on X simplicially. Let $Y \leq X$ be a subcomplex which is a strict fundamental domain for the action, that is, the complex Y intersects each G -orbit in exactly one point.*

We assume that Y is connected and simply connected, and we write $G_0 = \langle G_s : s \in Y \rangle$ for the subgroup generated by the stabilisers of simplices of Y . Then

$$\begin{aligned}\pi_0(|X|) &\cong G/G_0, \\ \pi_1(|X|) &\cong \ker(\varinjlim \{G_s : s \in Y\} \twoheadrightarrow G_0),\end{aligned}$$

where the direct limit is taken over the system of stabilisers $(G_s)_{s \in Y}$ with the natural inclusions.

We will use the language of simple complexes of groups and of stratified spaces in the proof. Since the definition of these objects would enlarge this paper unnecessarily, we will try to treat these concepts in a ‘black box’ fashion. All definitions and all the results we use can be found in [BH99, Chapter II.12].

Proof All references given in this proof and all missing definitions are contained in Chapter II.12 in [BH99]. The main result we need for this proof is Proposition 12.20. We have adopted the notation of this proposition here. In the following, we will construct the objects appearing there and we will relate our situation to the situation in the proposition.

The geometric realisation $|X|$ is a *stratified simplicial complex* in the sense of Definition 12.1 and the discussion thereafter, where the strata are given by the G -images of closed simplices of Y . The set of strata \mathcal{Q} for the subcomplex $|Y|$ is given by the closed simplices in Y .

The group G then acts by strata preserving morphisms on X , and we construct the associated *complex of groups* $G(\mathcal{Q})$ as in 12.15. By Proposition 12.20(2), we then have $\pi_0(|X|) \cong G/G_0$. If we consider the G_0 -action on the connected component $G_0 \cdot Y \subseteq X$, we obtain by Proposition 12.20(4) that

$$\pi_1(|X|) \cong \ker(\widehat{G(\mathcal{Q})} \twoheadrightarrow G_0),$$

where $\widehat{G(\mathcal{Q})} \cong \varinjlim \{G_s : s \in Y\}$ by Definition 12.12. \square

This can be used to calculate the number of connected components and the fundamental group of the geometric realisation of the Wagoner complex $|\mathcal{W}(G)|$.

Theorem 14 *Assume that G is a group of Kac-Moody type satisfying condition (Co*) and such that the associated root system is 2-spherical. Then we have*

$$\pi_0(|\mathcal{W}(G)|) \cong G/G^\dagger.$$

If G is of non-spherical type or of rank at least 3, we have

$$\pi_1(|\mathcal{W}(G)|) \cong \ker(\hat{G} \twoheadrightarrow G^\dagger).$$

Proof Consider the action of G on the Wagoner complex $\mathcal{W}(G)$. The stabiliser of a vertex (U_s) is obviously the group U_s itself, the stabiliser of a flag

$$(U_{s_1} \subset U_{s_2} \subset \cdots \subset U_{s_r})$$

is given by the smallest group U_{s_1} . The group $G_0 = \langle U_s : s \in \Sigma \rangle$ is equal to the little projective group G^\dagger . The direct limit of all stabilisers of simplices in \mathcal{A} with the natural inclusions is hence the group \tilde{G} .

A strict fundamental domain for the action on $|\mathcal{W}(G)|$ is given by the geometric realisation of the standard apartment $|\mathcal{A}|$. If G is of spherical type of rank n , then $|\mathcal{A}|$ is homotopy equivalent to a $(n-1)$ -sphere, and hence connected and simply connected if $n \geq 3$.

If G is of non-spherical type, then $|\mathcal{A}|$ is homeomorphic to the Davis realisation of the Coxeter system associated to G as defined in [Dav08, Chapter 7], which is contractible by Theorem 8.2.13 in [Dav08].

So we can apply Theorem 13 and use that we have $\tilde{G} \cong \hat{G}$ by Proposition 11. \square

Corollary 15 *In particular, if the prerequisites of Theorems 12 and 14 are satisfied, we have*

$$\pi_1(|\mathcal{W}(G)|) \cong H_2(G^\dagger; \mathbb{Z}).$$

6 The connection to algebraic K -theory

Wagoner complexes were originally introduced by Wagoner in [Wag73] to provide a definition of higher algebraic K -theory. He then proved the following theorem, which is a special case of our result only if R is a division ring.

Theorem 16 (Wagoner, Proposition 2 in [Wag73]) *Let R be a ring. If $n \geq 2$ then*

$$\pi_0(|\mathcal{W}(\mathrm{GL}_{n+1}(R))|) \cong \mathrm{GL}_{n+1}(R)/\mathrm{E}_{n+1}(R) \cong H_1(\mathrm{GL}_{n+1}(R); \mathbb{Z}) \cong K_1(R).$$

If $n \geq 3$, then

$$\pi_1(|\mathcal{W}(\mathrm{GL}_{n+1}(R))|) \cong \ker(\mathrm{St}_{n+1}(R) \twoheadrightarrow \mathrm{E}_{n+1}(R)) \cong H_2(\mathrm{E}_{n+1}(R); \mathbb{Z}) \cong K_2(R).$$

In subsequent papers [AKW73] and [AKW77], Anderson, Karoubi and Wagoner and in a final paper [Wag78] Wagoner proved that Wagoner's definition of higher algebraic K -theory coincides with Quillen's definition of K -theory in the sense that

$$K_i(R) \cong \pi_{i-1}(|\mathcal{W}(\mathrm{GL}_{n+1}(R))|) \quad \text{for } n \gg i, i > 0.$$

This leads naturally to the question whether

$$\pi_{i-1}(|\mathcal{W}(G)|) \cong \pi_i(BG^+)$$

for any group with $\mathrm{rank}(G) \gg i$. This question would be interesting for hermitian K -theory, for example, by choosing the group G to be a symplectic or orthogonal group. Unfortunately, we do not know whether these groups are isomorphic.

7 Affine Wagoner complexes

For completeness, we also include the following construction, imitating the corresponding construction in [Wag75]. Let G be a group with a spherical root datum with discrete valuation, compare [Wei09, Chapter 3]. Let Δ be the associated affine building and let Σ be a fixed apartment of Δ . We will denote the set of spherical roots by Φ and the half-apartments of Σ by $H_{\alpha,k}$ for $\alpha \in \Phi$, $k \in \mathbb{Z}$ with associated walls $\partial H_{\alpha,k}$.

Definition 17 *Let s be a simplex in Σ and let $n \geq 1$. The n -cell around s is defined to be the set*

$$C_n(s) := \bigcap_{\substack{\alpha \in \Phi, k \in \mathbb{Z} \\ R_s \subset H_{\alpha,k}}} H_{\alpha,k}.$$

Definition 18 *For any $n \geq 1$ and any simplex $s \in \Sigma$, we write*

$$U_s^n := \langle U_{\alpha,k} : C_n(s) \subset H_{\alpha,k}, \alpha \in \Phi, k \in \mathbb{Z} \rangle.$$

Analogously to Definition 2, for any $n \geq 1$, we define the affine Wagoner complex $\mathcal{W}^n(G)$ to be the flag complex over the partially ordered set of cosets $\{gU_s^n : s \in \Sigma, g \in G\}$.

For every fixed n , we have by construction $s \subseteq s'$ implying that $U_s^n \leq U_{s'}^n$. There are also simplicial epimorphisms from $\mathcal{W}^n(G)$ onto the barycentric subdivisions of certain ‘coarsenings’ of the affine building Δ . For buildings of type \tilde{A}_n , this can be found in [Wag75, §5].

Remark As above, we define the standard apartment of the Wagoner complex by

$$\mathcal{A}^n = \text{Flag}(\{gU_s^n : g \in G, s \in \Sigma\}).$$

Obviously, the apartment \mathcal{A}^n is isomorphic to the flag complex over the set of n -cells $\{C_n(s) : s \in \Sigma\}$. It is not hard to see that this complex is in turn isomorphic to the complex associated to the affine reflection group generated by the reflections at the walls $\partial H_{\alpha, nk}$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. In particular, by rescaling by $\frac{1}{n}$, this complex is isomorphic to the Coxeter complex Σ , so \mathcal{A}^n is contractible.

In analogy to the construction above, we denote the direct limit of the system of groups $\{U_s^n : s \in \Sigma\}$ with their canonical inclusions by \tilde{G}^n . By the same argumentation as in the proof of Theorem 14, using that \mathcal{A}^n is contractible, we obtain

Proposition 19 *For any $n \geq 1$, we have*

$$\pi_1(|\mathcal{W}^n(G)|) \cong \ker(\tilde{G}^n \twoheadrightarrow G^\dagger)$$

Construction If n divides m , then $C_n(s) \subseteq C_m(s)$ and hence $U_s^n \supseteq U_s^m$ for any $s \in \Sigma$. Consequently, the map $p_n^m : \mathcal{W}^m(G) \rightarrow \mathcal{W}^n(G)$ induced by $gU_s^m \mapsto gU_s^n$ is well-defined.

We obtain an inverse system of the simplicial complexes $\mathcal{W}^n(G)$ with the maps p_n^m for n dividing m . This induces an inverse system on the fundamental groups $\pi_1(|\mathcal{W}^n(G)|)$.

Definition 20 *The inverse limit of this system is denoted by*

$$\pi_1^{\text{aff}}(G) := \varprojlim_n \pi_1(|\mathcal{W}^n(G)|)$$

and is called the affine fundamental group of G .

The following theorem was already proved by Wagoner in [Wag75, §2] for $G = \text{SL}_k(\mathbb{F})$ over a local field \mathbb{F} in a very different way.

Theorem 21 *Let $\mathcal{W}(G)$ be the Wagoner complex associated to the spherical building at infinity. Assume that the rank of the spherical building is at least three. Then, for every $n \geq 1$, there is a homomorphism*

$$\pi_1(|\mathcal{W}(G)|) \rightarrow \pi_1(|\mathcal{W}^n(G)|)$$

that is compatible with the inverse system above. In particular, we obtain a homomorphism

$$\pi_1(|\mathcal{W}(G)|) \rightarrow \pi_1^{\text{aff}}(G).$$

Proof We will first construct a homomorphism $\hat{G} \rightarrow \tilde{G}^n$. By the universal property of the direct limit, it is enough to construct homomorphisms $U_\alpha, U_{[\alpha, \beta]} \rightarrow \tilde{G}^n$ which are compatible with the natural inclusions.

Note that by the definition of root data with valuations, we have $U_\alpha = \bigcup_{k \in \mathbb{Z}} U_{\alpha, k}$, this is a filtration and hence a direct limit. Again, it is enough to construct a homomorphism $U_{\alpha, k} \rightarrow \tilde{G}^n$ for all $k \in \mathbb{Z}$. The group $U_{\alpha, k}$ is contained in all groups U_s^n such that $C_n(s) \subseteq H_{\alpha, k}$. There are obviously such simplices, and we consider the inclusions $U_{\alpha, k} \rightarrow U_s^n$. By an argument similar to the one in the proof of Proposition 11, using that half-apartments are connected and that all homomorphisms in the system of groups are inclusions, these maps are compatible with the system of groups for \tilde{G}^n , and we obtain homomorphisms $U_{\alpha, k} \hookrightarrow \tilde{G}^n$.

For the groups $U_{[\alpha, \beta]}$, consider the filtration $U_{[\alpha, \beta], k} = \langle U_{\gamma, k} : \gamma \in [\alpha, \beta] \rangle$. Then the group $U_{[\alpha, \beta], k}$ is contained in all groups U_s^n with $C_n(s) \subseteq \bigcap_{\gamma \in [\alpha, \beta]} H_{\gamma, k}$. This intersection contains a sector and hence such simplices exist. It is also connected and by the same argument as before, the inclusion is compatible with the system of groups and we obtain a homomorphism $U_{[\alpha, \beta]} \rightarrow \tilde{G}^n$.

Since all of these maps are induced by inclusions, they are automatically compatible with the system of groups for \hat{G} . Hence we obtain a homomorphism $\hat{G} \rightarrow \tilde{G}^n$ which makes the following diagram commute:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(|\mathcal{W}(G)|) & \longrightarrow & \hat{G} & \longrightarrow & G^\dagger \longrightarrow 1 \\ & & \downarrow \text{dotted} & & \downarrow & & \parallel \\ 1 & \longrightarrow & \pi_1(|\mathcal{W}^n(G)|) & \longrightarrow & \tilde{G}^n & \longrightarrow & G^\dagger \longrightarrow 1. \end{array}$$

For standard reasons, the dotted map completing the above diagram exists. It is the desired homomorphism. \square

In the case of $G = \mathrm{SL}_k(\mathbb{F})$ over a local field \mathbb{F} , Wagoner proves in [Wag75], that $\pi_1^{\mathrm{aff}}(G)$ is isomorphic to the kernel of the universal central topological extension of $G = G^\dagger = \mathrm{SL}_k(\mathbb{F})$, sometimes denoted by $H_2^{\mathrm{top}}(\mathrm{SL}_k(\mathbb{F}))$. We do not know whether this is true in the general case. Wagoner's proof relies heavily on the fact that $H_2(\mathrm{SL}_k(\mathbb{F})) \cong H_2^{\mathrm{top}}(\mathrm{SL}_k(\mathbb{F})) \oplus D$, where D is infinitely divisible, and that $H_2^{\mathrm{top}}(\mathrm{SL}_k(\mathbb{F}))$ is isomorphic to the group of roots of unity of \mathbb{F} .

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